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Hole-vortex solitons

Aleksandr N. Bugay*
Department of Theoretical Physics, I. Kant State University of Russia, 236041 Kaliningrad, Russia

Sergey V. Sazonov†
Russian Research Centre Kurchatov Institute, 123182 Moscow, Russia

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Three-dimensional two-component solitons, propagating in long-short wave resonance mode, are predicted. If the spectrum of the short-wave component lies in the area of normal group velocity dispersion, these solitons have transverse structure in the form of hole-vortex field defects on an infinite background. In the opposite case two-component "bullets" or the "bright" vortex and the "bullet" with a hole in the center can exist. The stability region of the considered objects is estimated on the basis of a variational approach. As a concrete physical model we consider the propagation of electromagnetic pulses in a uniaxial crystal. Here the ordinary component of the pulse is the short wave, and its extraordinary component is the long wave.

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I. INTRODUCTION

In recent decades investigation of three-dimensional localized wave objects has become one of the most important directions in the area of nonlinear optics. There have been found such spatiotemporal structures as optical bullets [1,2], X waves (or X solitons) [3], vortex solitons [4,5], etc. All these achievements have been obtained using the slowly varying envelope approximation (SVEA). Such a traditional approach is described in detail in the monographic literature [6,7].

The development of laser optics has involved producing increasingly shorter pulses, down to few-cycle pulses [8]. In this case the SVEA is not applicable, because such pulses have a very broad spectrum. Therefore approaches have been developed with model equations, which have been derived directly for the pulse field [8–13]. The limiting case of few-cycle pulses is the pulse in a half period of oscillation—i.e., presenting itself as a single hump without carrier frequency. We will call such an object an ultimately short pulse (USP).

It is well known that spatiotemporal localization is a result of the competition between nonlinear self-steepening and dispersive-diffractive effects. It is important to notice a specific aspect of the USP. The relative role of diffraction effects in the case of quasimonochromatic pulses can be expressed by the relation λ/R, where $\lambda=2\pi c/\omega$ is the wavelength, ω is the carrier frequency, c is the speed of light in vacuum, and R is the characteristic transverse size of the pulse. In the case of the USP the wavelength should be replaced by a pulse longitudinal size $l_{\parallel}=v_g\tau_p$, where τ_p is the pulse temporal duration and v_g is the group velocity.

The general theory of nonlinear waves admits the existence of special localized objects, which consist of both an envelope (quasimonochromatic) soliton and USP soliton. For example, it may be the model of the nonlinear Shrödinger equation (NLSE) with temperature-dependent refractive index [14].

In the present paper we examine another similar system, first obtained in plasmas [15,16], but recently found in a process of nonlinear pulse propagation in dielectrics [17] and also in phenomena, connected with electromagnetically induced transparency [18].

Let us consider the nonlinear propagation of an optical pulse in uniaxial crystal more closely. This process is accompanied by the formation of two-component solitons. The input ordinary quasimonochromatic pulse during the propagation generates an extraordinary wave in the form of a USP soliton and experiences scattering of it. The ordinary component obtains the shift Ω of the carrier frequency in the red spectral range. The value of this shift is proportional to the intensity of the component considered. The self-scattering process mentioned is effective only if the condition of Zakharov-Benney resonance is fulfilled [15]. According to this condition, the group velocity v_g of the ordinary component is equal to the phase velocity v_{ph} of the extraordinary component.

However, the transverse structure of the solitons mentioned in more than one space dimension has currently not been explored in detail. Taking into account the difference discussed in transverse effects in case of quasimonochromatic and USP components it is possible to expect new features in process of space-time localization. Thus, the goal of the present work is to find out what types of three-dimensional structures are possible.

The paper is organized as follows. In Secs. II and III we derive three-dimensional nonlinear wave equations and obtain their approximate stationary solutions under the condition of different signs of group velocity dispersion (GVD). The analysis is based on the variational approach. In Sec. IV we consider the stability of solutions obtained and estimate their existence region. Finally, in Sec. V we summarize our results and give the conclusions.

II. MODEL EQUATIONS

Let us consider nonlinear propagation of optical pulses in uniaxial crystal. We assume that the initial laser pulse polar-

*Electronic address: bugay@newmail.ru
†Electronic address: barab@newmail.ru

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ization coincides with the polarization of ordinary waves. The propagation occurs along the z axis, which is perpendicular to the optical axis. The wave equation has form

$$\Delta \mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 \mathbf{P}}{\partial t^2}, \tag{1}$$

where $\mathbf{E}$ is the electric field and $\mathbf{P}$ is the macroscopic polarization.

Henceforth, we will assume that diffraction effects are weak (paraxial approximation) and the field components depend mainly on z and t. We also consider that the spectrum of the pulse lies in the transparency region of the crystal. Therefore, the dispersion and nonlinear effects are relatively small. In this case the polarization can be presented as a sum $\mathbf{P}=\mathbf{P}_{lin}+\mathbf{P}_{nl}$ of linear term $\mathbf{P}_{lin}$ and term $\mathbf{P}_{nl}$, including the effects of nonlinearity and dispersion. From the Maxwell equations it follows that $\nabla\cdot\mathbf{D}=\nabla\cdot(\mathbf{E}+4\pi\mathbf{P})=0$. From here we make an estimation $\nabla\cdot\mathbf{E}=-4\pi\nabla\cdot\mathbf{P}_{nl}/(1+4\pi\chi_l)$, where χ_l is the linear inertialless susceptibility. The second term in Eq. (1) is small by reason of the smallness of both nonlinear parts of the medium response and diffraction effects. Therefore, we neglect this term.

Thus, we obtain the following set of equations:

$$\Delta E_{o,e} - \frac{1}{c^2}\frac{\partial^2 E_{o,e}}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 P_{o,e}}{\partial t^2}, \tag{2}$$

where E_o is the ordinary pulse field component and E_e is its extraordinary component.

We write corresponding polarizations in the form [17]

$$P_o = \int_0^\infty \tilde{\chi}_o(t')E_o(\mathbf{r},t-t')dt' + 2\chi_{eo}E_eE_o, \tag{3}$$

$$P_e = \int_0^\infty \tilde{\chi}_e(t')E_e(\mathbf{r},t-t')dt' + \chi_{ee}E_e^2 + \chi_{eo}E_o^2, \tag{4}$$

where $\tilde{\chi}_o(t)=\chi_{xx}^{(1)}(t)$ and $\tilde{\chi}_e(t)=\chi_{yy}^{(1)}(t)$ are the ordinary and extraordinary components of linear electronic susceptibility tensor taking into account the time delay of the response (dispersion). The coefficients $\chi_{eo}=\chi_{xxy}^{(2)}$ and $\chi_{ee}=\chi_{yyy}^{(2)}$ are the nonlinear inertialless susceptibilities of second order.

Let us comment on expressions (3) and (4). They reflect the symmetry properties of the uniaxial medium. Since the invariance with respect to spatial reflections is normal to the optical axis, expressions (3) and (4) are invariant with respect to transformations $P_o \to -P_o$, $E_o \to -E_o$, but the invariance with respect to transformation $P_e \to -P_e$, $E_e \to -E_e$ is violated.

Let us consider that the ordinary field component is the quasimonochromatic pulse. This means that the spectral width $\delta\omega \sim \tau_p^{-1}$ of this component is much smaller than its carrier frequency ω—i.e., $\omega\tau_p \gg 1$. In this case the field of ordinary components can be written in the form of an envelope pulse

$$E_o(\mathbf{r},t) = \xi(\mathbf{r},t)\exp[i(\omega t - kz)] + \text{c.c.}, \tag{5}$$

where $\xi(\mathbf{r},t)$ is the slowly varying envelope, ω is the carrier frequency, and k is the wave number.

Using the slow variation of envelope ξ at time scale t', we perform an expansion into a Taylor series in t':

$$\int_0^\infty \tilde{\chi}_o(t')E_o(\mathbf{r},t-t')dt' = \left[\chi_o(\omega)\xi - i\left(\frac{\partial \chi_o}{\partial \omega}\right)\frac{\partial \xi}{\partial t} - \frac{1}{2}\left(\frac{\partial^2 \chi_o}{\partial \omega^2}\right)\frac{\partial^2 \xi}{\partial t^2}\right]\exp[i(\omega t - kz)] + \text{c.c.}, \tag{6}$$

where the frequency susceptibility is

$$\chi_o(\omega) = \int_0^\infty \tilde{\chi}_o(t')e^{-i\omega t'}dt'.$$

As follows from Eq. (2) and expressions (3) and (4) the extraordinary component can be centered at doubled frequency 2ω or zero frequency. The first case corresponds to a well-known effect of second-harmonic generation. But here we are mostly interested in the second one, which is the generation of the USP (video pulse).

The integrand in Eq. (4) can be expanded in t' due to the relative smallness of dispersion effects:

$$\int_0^\infty \tilde{\chi}_e(t')E_e(\mathbf{r},t-t')dt' = \chi_e(0)E_e(\mathbf{r},t) - \chi_e'(0)\frac{\partial E_e(\mathbf{r},t)}{\partial t} - \frac{\chi_e''(0)}{2}\frac{\partial^2 E_e(\mathbf{r},t)}{\partial t^2}, \tag{7}$$

where

$$\chi_e(0) = \int_0^\infty \chi_e(t')dt'$$

is the inertialless linear susceptibility and

$$\chi_e'(0) = \int_0^\infty t'\chi_e(t')dt', \quad \chi_e''(0) = -2\int_0^\infty t'^2\chi_e(t')dt'.$$

Here we can neglect the term $\sim\partial E_e/\partial t$, which describes the decay of the polarization response of a medium caused by irreversible relaxation. The characteristic time T_2 of this process is contained in the constant $\chi_e'(0)$. The condition $\tau_p \ll T_2$ is fulfilled with good accuracy for picosecond pulses and a broad class of crystals.

By substituting Eqs. (3) and (7) into Eq. (2) and neglecting rapidly oscillating terms, we obtain

$$i\left(\frac{\partial \xi}{\partial z} + \frac{1}{v_g}\frac{\partial \xi}{\partial t}\right) + \frac{k_2}{2}\frac{\partial^2 \xi}{\partial \tau^2} = \alpha_{eo}\xi E_e + \frac{c}{2n\omega}\Delta_\perp \xi, \tag{8}$$

$$\frac{\partial^2 E_e}{\partial z^2} - \frac{n_e^2}{c^2}\frac{\partial^2 E_e}{\partial t^2} = \frac{\partial^2}{\partial t^2}(\alpha_{oe}|\xi|^2 + \alpha_{ee}E_e^2) - \delta_e \frac{\partial^4 E_e}{\partial t^4} - \Delta_\perp E_e, \tag{9}$$

where $v_g = c/[n+\omega(\partial n/\partial\omega)]$ is the group velocity of the ordinary component, $n=\sqrt{1+4\pi\chi_o(\omega)}$ and $n_e=\sqrt{1+4\pi\chi_e(0)}$ are the refractive indices of ordinary and extraordinary components, respectively, $k_2=\partial v_g^{-1}/\partial\omega$ is the parameter of GVD, and $\Delta_\perp$ is the transverse Laplacian. The constants $\alpha_{eo} = 4\pi\chi_{eo}\omega/c$, $\alpha_{oe}=8\pi\chi_{ee}/c^2$, and $\alpha_{ee}=4\pi\chi_{eo}/c^2$ describe the nonlinear effects, and $\delta_e = 2\pi\chi_e''(0)/c^2$ is the dispersion parameter of the extraordinary component.

It is visible from Eqs. (8) and (9) that the ordinary wave plays a dominant role in pulse propagation. If the initial pulse contains only ordinary waves, it can generate the video pulse of extraordinary waves. The inverse process is forbidden. We should note that the interaction between pulse components is the most effective when the condition of Zakharov-Benney long-short wave resonance is fulfilled [15]. This means that the group velocity of the ordinary component is equal to the phase velocity of the extraordinary component:

$$v_g = c/n_e. \tag{10}$$

The ordinary wave also makes the main contribution to dispersion, so we neglect the small term $\sim\partial^4 E_e/\partial t^4$ in Eq. (9). Because of the smallness of the nonlinear and transverse effects on the right-hand side of Eq. (9), we can neglect the reflected wave and apply the approximation of unidirectional propagation. Summarizing this along with condition (10), we finally arrive at the following set of equations:

$$i\frac{\partial\xi}{\partial z} + \frac{k_2}{2}\frac{\partial^2\xi}{\partial\tau^2} - a\omega E_e\xi = \frac{c}{2n\omega}\Delta_\perp\xi, \tag{11}$$

$$\frac{\partial E_e}{\partial z} + \frac{\partial}{\partial\tau}(a|\xi|^2 + bE_e^2) = \frac{c}{2n}\Delta_\perp\int_{-\infty}^{\tau} E_e d\tau', \tag{12}$$

where $\tau=t-z/v_g$ is the "local" time, $a=4\pi\chi_{eo}/(n_e c)$, and $b = 2\pi\chi_{ee}/(n_e c)$.

In the one-dimensional case under condition $b=0$, when the right-hand sides of Eqs. (11) and (12) are equal to zero, this system transforms into the Yajima-Oikawa equations [16], which are a unidirectional version of the Zakharov equations [15]. The Yajima-Oikawa system is integrable by the method of inverse scattering transformation.

Our generalization of the integrable system mentioned includes transverse effects (diffraction) and quadratic nonlinearity, owned by the USP component. The last one is rather small [17], and we will account for it as a perturbation.

III. VARIATIONAL APPROACH

The influence of transverse effects on the solitons of system (11), (12) can be taken into account with the help of the method of averaged Lagrangians [19–22].

The system (11), (12) has one-dimensional solitonlike solutions. We write them as an expansion of the small parameter:

$$\varepsilon = \frac{8b}{3a}\frac{\omega}{\Omega}(\omega\tau_p)^{-2} \sim \frac{bE_e^2}{a|\xi|^2} \ll 1.$$

Assuming $a\sim b$, $\omega/\Omega\sim 10^4$ [17], and $\tau_p\sim 1$ ps we find $\varepsilon \sim 10^{-1}-10^{-2}$. Thus, the one-dimensional solutions, obtained with $O(\varepsilon^2)$ accuracy, have the form

$$\xi = \xi_m \operatorname{sech}\eta\left(1 + \frac{\varepsilon}{4} - \frac{\varepsilon}{8}\operatorname{sech}^2\eta\right)e^{-i(\Omega\tau+\Phi)}, \tag{13}$$

$$E_e = -E_m \operatorname{sech}^2\eta\left(1 + \frac{\varepsilon}{8} - \frac{5\varepsilon}{16}\operatorname{sech}^2\eta\right). \tag{14}$$

Here

$$\xi_m = \frac{|k_2|}{a\tau_p}\sqrt{\frac{\Omega}{\omega}}, \quad E_m = \frac{k_2}{a\omega\tau_p^2}, \quad \eta = \frac{(\tau + k_2\Omega z)}{\tau_p},$$

$$\Phi = 0.5k_2(\Omega^2 - \tau_p^{-2})z.$$

We should note that this solution has two free parameters: the duration τ_p and the frequency shift Ω. We can also present this set of parameters as duration and intensity, because Ω is proportional to the intensity of the ordinary component, $I_o \approx c\xi_m^2/4\pi$ [see Eq. (13)].

Putting $\tau_p^{-1}\to\theta(\mathbf{r})$ and $\Phi\to\Phi(\mathbf{r})$ into Eqs. (13) and (14), we obtain the three-dimensional approximate solutions

$$\xi = \xi_m\tau_p\theta(\mathbf{r})\operatorname{sech}\eta' e^{-i[\Omega\tau+\Phi(\mathbf{r})]}\left(1 + \frac{1}{4}\varepsilon\tau_p^2\theta^2(\mathbf{r}) - \frac{1}{8}\varepsilon\tau_p^2\theta^2(\mathbf{r})\operatorname{sech}^2\eta'\right), \tag{15}$$

$$E_e = -E_m\tau_p^2\theta^2(\mathbf{r})\operatorname{sech}^2\eta'\left(1 + \frac{1}{8}\varepsilon\tau_p^2\theta^2(\mathbf{r}) - \frac{5}{16}\varepsilon\tau_p^2\theta^2(\mathbf{r})\operatorname{sech}^2\eta'\right), \tag{16}$$

where $\eta' = \theta(\mathbf{r})(\tau + k_2\Omega z)$.

The new functions θ and Φ define the inverse temporal duration of the soliton and the eikonal of the quasimonochromatic component, respectively. For the purpose of correlation between three-dimensional and one-dimensional solutions we assume hereafter that $\tau_p^{-1} = \max[\theta(z=0,x,y)]$.

The Lagrangian density, corresponding to Eqs. (11) and (12), is given by the expression

$$\mathcal{L} = \frac{i}{2}\left(\xi\frac{\partial\xi^*}{\partial z} - \xi^*\frac{\partial\xi}{\partial z}\right) + \frac{k_2}{2}\left|\frac{\partial\xi}{\partial\tau}\right|^2 + a\omega|\xi|^2\frac{\partial Q}{\partial\tau} - \frac{c}{2n\omega}|\nabla_\perp\xi|^2 + \frac{\omega}{2}\frac{\partial Q}{\partial z}\frac{\partial Q}{\partial\tau} - \frac{c\omega}{2n}(\nabla_\perp Q)^2 + \frac{\omega b}{3}\left(\frac{\partial Q}{\partial\tau}\right)^3. \tag{17}$$

The function Q is connected with the field of the extraordinary component by the relation $E=\partial Q/\partial\tau$.

After substitution of Eqs. (15) and (16) into Eq. (17) and integrating over the temporal variable τ, we find the averaged Lagrangian

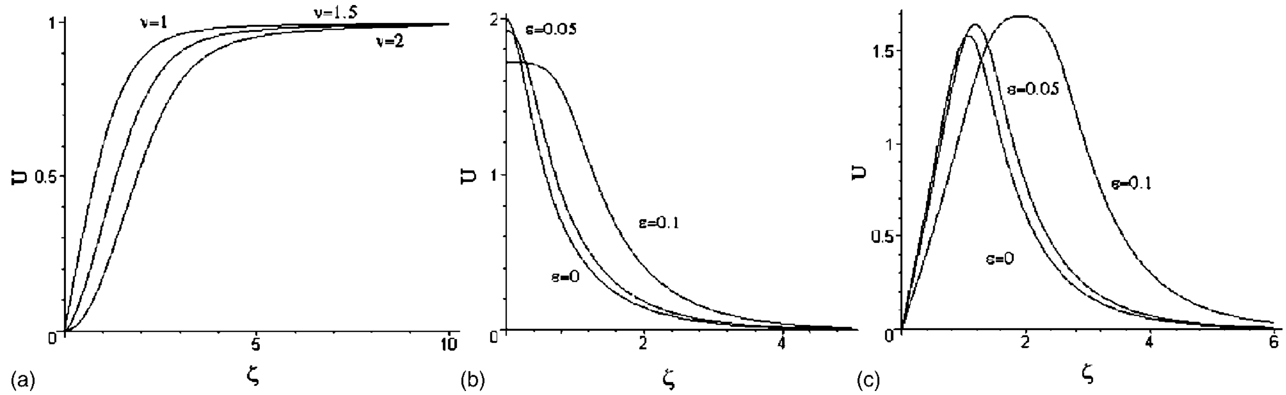

FIG. 1. The solutions of Eq. (25) in a region of normal GVD with different ν (a) and in a region of anomalous GVD with $\nu=0$ (b) and $\nu=1$ (c).

$$\Lambda = \rho\left(\frac{\partial\Phi}{\partial z} + \frac{c}{2n\omega}(\nabla_\perp\Phi)^2\right) + \frac{1}{6}k_2\rho^3 - \frac{1}{10}\varepsilon\tau_p^2 k_2\rho^5 - \frac{1}{2}k_2\Omega^2\rho + \frac{c}{8n\omega\nu_0^2}(1+\gamma\varepsilon\tau_p^2\rho^2)\frac{(\nabla_\perp\rho)^2}{\rho}, \tag{18}$$

where

$$\rho = \theta + \frac{\varepsilon\tau_p^2}{3}\theta^3, \quad \nu_0 = 3\left[\pi^2 + 12 + \frac{\omega}{\Omega}(\pi^2-6)\right]^{-1/2},$$

$$\gamma = \frac{4}{9}\nu_0^2\left(\frac{\pi^2}{15} + 2 + \frac{\omega}{2\Omega}\right).$$

Using Eq. (18) to write out the Euler-Lagrange equations for functions ρ and Φ, we get

$$\frac{\partial\rho}{\partial z} + \frac{c}{n\omega}\nabla_\perp(\rho\nabla_\perp\Phi) = 0, \tag{19}$$

$$\frac{\partial\Phi}{\partial z} + \frac{c}{2n\omega}(\nabla_\perp\Phi)^2 + \frac{k_2}{2}(\rho^2 - \varepsilon\tau_p^2\rho^4 - \Omega^2) = \frac{c}{2n\omega\nu_0^2}\left[(1-\gamma\varepsilon\tau_p^2\rho^2)\frac{\Delta_\perp\sqrt{\rho}}{\sqrt{\rho}} - 2\gamma\varepsilon\tau_p^2\rho(\nabla_\perp\sqrt{\rho})^2\right]. \tag{20}$$

The system of Eqs. (19) and (20) obtained can be presented as a single equation for the complex function

$$\psi = \sqrt{\rho}\exp[i(0.5k_2\Omega^2 z - \Phi)]. \tag{21}$$

The corresponding equation looks like

$$i\frac{\partial\psi}{\partial z} - \frac{c}{2n\omega}\Delta_\perp\psi + \frac{1}{2}k_2(|\psi|^4\psi - \varepsilon\tau_p^2|\psi|^8\psi) = \frac{c}{2n\omega\nu_0^2}\hat{D}_\perp\psi, \tag{22}$$

where the nonlinear operator $\hat{D}_\perp$ is defined as follows:

$$\hat{D}_\perp\psi = (1-\nu_0^2)\frac{\psi}{|\psi|}\Delta_\perp|\psi| - \gamma\varepsilon\tau_p^2|\psi|^3\psi\Delta_\perp|\psi| - 2\gamma\varepsilon\tau_p^2|\psi|^2\psi(\nabla_\perp|\psi|)^2. \tag{23}$$

Thus, the averaged Lagrangian method can be used to reduce the analysis of three-dimensional spatiotemporal solitons to a solution of Eq. (22), which describes the evolution of soliton parameters. Similar equations are used for a description of the nonlinear dynamics of a spatial beam. An important feature here is the presence of competing nonlinearities on the left-hand side of Eq. (22). In the case of anomalous GVD ($k_2<0$) this often results in an arrest of the collapse [6,7].

Let us find stationary solutions of Eq. (22). The corresponding solution can be written as

$$\psi = \tau_p^{-1/2}U(r)\exp[i(qz + m\varphi)], \tag{24}$$

where r and φ are the radial and angular components of the cylindric coordinate system, q is the nonlinear shift of the wave vector, and $m=0,\pm1,\pm2,\ldots$ is the topological charge (vorticity) of soliton.

After simple transformations we obtain that the amplitude U obeys the following equation:

$$(1-\varepsilon\gamma U^4)\left(\frac{d^2U}{d\zeta^2} + \frac{1}{\zeta}\frac{dU}{d\zeta}\right) - 2\varepsilon\gamma U^3\left(\frac{dU}{d\zeta}\right)^2 - \frac{\nu^2}{\zeta^2}U + \text{sgn}(k_2)\times(U - U^5 + \varepsilon U^9) = 0, \tag{25}$$

where $\zeta=r/R_s$, $R_s=c\tau_p(\nu_0^2 cn\omega|k_2|)^{-1/2}$, $\nu=\nu_0|m|$, and $q=q_s=k_2/2\tau_p^2$.

Solving this equation we can find unknown functions in approximate solutions (15) and (16). They are given by the formulas

$$\theta = \tau_p^{-1}U^2(r)\left(1 - \frac{\varepsilon}{3}U^4(r)\right), \tag{26}$$

$$\Phi = \frac{k_2}{2}(\Omega^2 - \tau_p^{-2})z + m\varphi. \tag{27}$$

In case of normal GVD ($k_2>0$) the only possible localized solutions are "dark" vortices on an infinite background. The solutions mentioned can be obtained by solving Eq. (25) numerically with boundary conditions $U(0)=0$, $U(\infty)=U_\infty \neq 0$, where $U_\infty \approx 1+\varepsilon/4$. The results are displayed in Fig. 1(a).

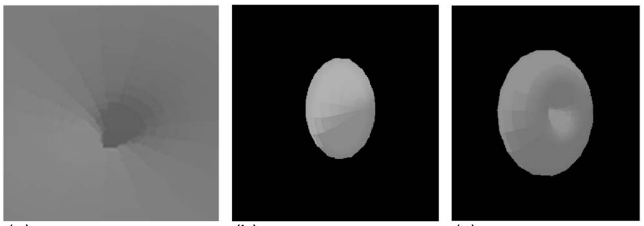

FIG. 2. An illustration of the field distribution [see approximate solutions (15) and (16), with account for expressions (26) and (27)] in the case of defect (a), light "bullet" (b), and "bright" vortex (c).

The transverse structure of the ordinary component is a "dark" vortex with zero field at the z axis. The wave vector moves on the spiral, which is directed along the z axis. The extraordinary component propagates synchronically with the ordinary component. But instead of a vortex the last one has a "hole" in the center. For this reason such a two-component object under the condition $k_2>0$ can be named a "dark-hole–vortex soliton." In reality any pulse has finite transverse size; therefore, such objects may exist like some kind of field defects on a quasi-one-dimensional pulse [see Fig. 2(a)].

Let us consider the situation with anomalous GVD—i.e., $k_2<0$. In this case solutions, localized in the transverse direction, are possible. The boundary conditions here correspond to the fact that the function $U(\zeta)$ and its derivatives tend to zero while the coordinate ζ tends to infinity. Some solutions of the equation mentioned are shown in Figs. 1(b) and 1(c).

The solution without a hole in the center ($\nu=m=0$) corresponds to a two-component light "bullet" [see Fig. 2(b)]. If ν, $m\neq 0$, the ordinary component has the localized structure in form of a "bright" vortex. The field distribution has the form of a torus as is shown in Fig. 2(c). Starting from the analogy from the case of positive GVD, we can call these two-component localized objects with vortex structure "bright hole-vortex solitons."

The asymptotic behavior of the solutions discussed can be found directly from Eq. (25). Thus, we have near the center of solutions with nonzero vorticity $U\sim\zeta^{\nu}$. In the case of normal GVD this function tends to a nonzero background value, while in the opposite case it vanishes exponentially.

We should note that both types of solutions exist only if the value of ε is lower than $\varepsilon_{\max}$. In case of "dark" vortices $\varepsilon^{(+)}_{\max}\approx 0.25$, while for localized solutions in the region with anomalous GVD it is lowered to $\varepsilon^{(-)}_{\max}\approx 0.137$. This situation is typical for equations with competing nonlinearities [23], where a parameter of solutions lies in a limited area. We will discuss this question further in Sec. IV.

As follows from the analysis, the vortex solutions can be singular near the center. This means that transverse derivations of ρ tend to infinity at the point $r=0$. However, the proper solutions must be smooth. Thus, the two first derivatives $\partial\rho/\partial r$ and $\partial^2\rho/\partial r^2$ should be finite. Using the asymptote $\rho\sim r^{2\nu}$, we find a restriction $\nu\geq 1$, which means $m\geq m_{cr}=1/\nu_0$ (m also must be an integer). Thus, we have a restriction on the topological charge, which depends on the intensity of the ordinary component, because $\nu_0=\nu_0(\Omega)$ and $\Omega\sim I_o$.

Let us estimate some characteristic parameters in the stationary solutions. Assuming $\Omega/\omega\sim 10^{-4}$, we obtain that $\nu_0\sim 10^{-2}$ and $m_{cr}\sim 10^2$. Using the fact that $k_2\sim 1/\omega v_g\sim 1/\omega c$, we get $R_s\sim\nu_0^{-1}l_{\parallel}$, where $l_{\parallel}=c\tau_p$ is the characteristic longitudinal size of the pulse. Assuming $\tau_p\sim 1$ ps we find $R_s\sim 1$ mm.

IV. STABILITY OF STATIONARY SOLUTIONS

An important instability in the case of localized three-dimensional solutions is the wave collapse, which occurs during the self-focusing process. We apply the variational method of averaged Lagrangians again to estimate the stability region of the solutions obtained. This approach can correctly predict the collapse of pulses. However, the corresponding results are not clear when the weak azimuthal instability is considered. The last one essentially occurs for vortices with large topological charges. Per our suggestion the instability caused by azimuthal perturbations may be included implicitly. As follows from the comparison between variational and linear stability analysis (the model of the NLSE with cubic-quintic nonlinearities; see, e.g., [23]) the variational approach gives a more or less acceptable prediction for the stability region in the case of small topological charges. In our case the variational method is more simple and useful, so we restrict ourselves to this approximate analysis.

We start from the Lagrangian density, corresponding to Eq. (22). It looks like

$$\mathcal{L}_{\psi}=\frac{i}{2}\left(\psi\frac{\partial\psi^*}{\partial z}-\psi^*\frac{\partial\psi}{\partial z}\right)-\frac{c}{2n\omega}|\nabla_{\perp}\psi|^2-\frac{k_2}{6}|\psi|^6+\frac{k_2\varepsilon\tau_p^2}{10}|\psi|^{10}-\frac{c}{2n\omega\nu_0^2}\left(1+\gamma\varepsilon\tau_p^2|\psi|^4\right)(\nabla_{\perp}|\psi|)^2. \tag{28}$$

The general form of the trial solution is given by the expression

$$\psi=\tau_p^{-1/2}A(z)f(s)\exp\{i[q(z)+\sigma(z)r^2+m\varphi]\}, \tag{29}$$

where $s=r/R(z)$.

The parameters A, R, σ, and q correspond to the amplitude, radius, chirp, and wave vector shift, respectively. The function f may be selected in any form which approximates the exact solution of Eq. (22). After substitution of Eq. (29) into the Lagrangian density (28) we need to integrate it over transverse variables. Thus, we obtain the averaged Lagrangian $\Lambda_{\psi}=2\tau_p^3k_2^{-1}\int_0^{\infty}\mathcal{L}_{\psi}rdr$ in the form

$$\Lambda_{\psi}=J_1A^2R^2\frac{q'}{|q_s|}+J_2A^2R^4\left(\frac{\sigma'}{|q_s|}-4\nu_0^2R_s^2\sigma^2\right)-\mathrm{sgn}(k_2)\times(J_3A^6R^2-\varepsilon J_4A^{10}R^2)-J_5A^2R_s^2-\varepsilon\gamma J_6A^6R_s^2. \tag{30}$$

Here J_1–J_6 are positive constants:

$$J_1=\int_0^{\infty}sf^2(s)ds,\quad J_2=\int_0^{\infty}s^3f^2(s)ds,$$

$$J_3=\frac{1}{3}\int_0^{\infty}sf^6(s)ds,\quad J_4=\frac{1}{5}\int_0^{\infty}sf^{10}(s)ds,$$

$$J_5 = \nu^2 \int_0^\infty s^{-1} f^2(s) ds + \int_0^\infty s[f'(s)]^2 ds,$$

$$J_6 = \int_0^\infty s f^4(s)[f'(s)]^2 ds.$$

We start our analysis from consideration of solutions in the form of field defects, which exist in the area of normal GVD ($k_2 > 0$). We present the corresponding trial function $f(s)$ in the following form:

$$f(s) = \frac{s^\nu}{(1+s^2)^{\nu/2}}. \quad (31)$$

Assuming $A=\text{const}$, we neglect the influence of an infinite background, focusing attention on the properties of solution.

Though the integrals J_1–J_6 are infinite, we can apply a renormalization procedure to Eq. (30) using a presentation of the β function through the γ functions, and a property of the gamma function $\Gamma(x+1)=x\Gamma(x)$. Thus, we obtain

$$\Lambda_\psi = C_\infty\left[\frac{q'}{|q_s|}A^2R^2 - \frac{\nu+1}{2}\left(\frac{\sigma'}{|q_s|} - 4\nu_0^2 R_s^2 \sigma^2\right)A^2R^4 - \frac{\Gamma(3\nu)}{\Gamma(\nu)}A^6R^2 + \varepsilon\frac{\Gamma(5\nu)}{\Gamma(\nu)}A^{10}R^2\right] + \text{const},$$

where the infinite constant is $C_\infty = 0.5\nu\Gamma(-1)$.

Using the averaged Lagrangian to write out the Euler-Lagrange equations in q, σ, and R, we get the following expressions:

$$\frac{dR^2}{dz} = 0, \quad (32)$$

$$\sigma = -(2\nu_0^2|q_s|R_s^2 R)^{-1}\frac{dR}{dz} = 0, \quad (33)$$

$$q' = |q_s|\left(\frac{\Gamma(3\nu)}{\Gamma(\nu)}A^4 - \varepsilon\frac{\Gamma(5\nu)}{\Gamma(\nu)}A^8\right). \quad (34)$$

Thus, this type of solution is a stationary object, which is stable with respect to self-focusing.

Using the fact that for an exact solution $q' = |q_s|$, we can find from Eq. (34) the expression for the amplitude:

$$A^4 = \frac{5\Gamma(3\nu+1)}{6\Gamma(5\nu+1)\varepsilon}\left(1 - \sqrt{1 - \frac{\varepsilon}{\varepsilon_{\max}^{(+)}}}\right), \quad (35)$$

where

$$\varepsilon_{\max}^{(+)} = \frac{5\Gamma(3\nu+1)^2}{36\Gamma(5\nu+1)\Gamma(\nu+1)}. \quad (36)$$

The difference between analytic and numeric values of $\varepsilon_{\max}^{(+)}$ is significant, but the qualitative result for the dependence $A(\varepsilon)$ is right.

We now turn to the opposite case of anomalous GVD. From Eq. (30) we get four Euler-Lagrange equations for the parameters q, σ, R, and A. The first equation

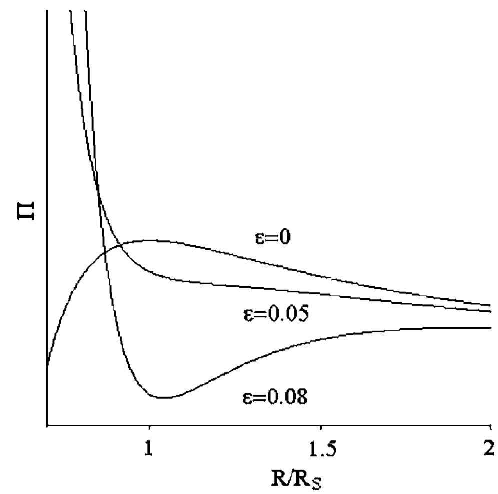

FIG. 3. The curve $\Pi(R)$ obtained numerically in case of a light "bullet" ($\nu=0$) with different values of ε.

$$\frac{d(A^2R^2)}{dz} = 0 \quad (37)$$

gives us the connection between the radius and amplitude:

$$A(z) = A_0\frac{R_0}{R(z)}, \quad (38)$$

where A_0 and R_0 are the initial values. Using that we obtain from the second equation

$$\sigma = -(4\nu_0^2|q_s|R_s^2R)^{-1}\frac{dR}{dz}. \quad (39)$$

After simple transformations we find the following equations for q and R:

$$q' = \frac{J_5|q_s|R_s^2}{J_1}\left(\frac{2}{R^2} - \frac{5\beta_1}{2R^4} + \varepsilon\frac{2\beta_2}{R^6} + \varepsilon\frac{9\beta_3}{4R^8}\right), \quad (40)$$

$$R'' = -\frac{\partial\Pi}{\partial R}, \quad (41)$$

where

$$\Pi = \frac{2\nu_0^2 J_5 q_s^2 R_s^4}{J_2}\left(\frac{1}{R^2} - \frac{\beta_1}{2R^4} + \varepsilon\frac{\beta_2}{3R^6} + \varepsilon\frac{\beta_3}{4R^8}\right). \quad (42)$$

Here $\beta_1 = 2J_3A_0^4R_0^4/J_5R_s^2$, $\beta_2 = 3\gamma J_6A_0^4R_0^4/J_5$, and $\beta_3 = 4J_4A_0^8R_0^8/J_5R_s^2$.

As follows from Eq. (41), the evolution of the radius and, consequently, other parameters can be driven from the analogy with the motion of Newtonian particle in an external field.

Let us apply the approximation

$$f(s) = s^\nu \exp(-s^2/2) \quad (43)$$

to calculate the parameters of the stationary solution. In this case $q' = -|q_s|$, $R=R_0$, $A=A_0$, and we also should imply two conditions $R''=0$ and $\partial^2\Pi/\partial R^2 > 0$. The first one demands the solution be stationary. The second one is the requirement of its stability, which means that the parameters obtained must correspond to a minimum of the potential (42) (see Fig. 3).

Using Eqs. (40)–(42) we obtain the following expressions for the amplitude,

$$A^4 = \frac{3^{3\nu}\Gamma(\nu+1)[10-3(\nu+1)\mu\delta(R_s/R)^2]}{\Gamma(3\nu+1)} \times \left(1-\sqrt{1-\frac{72\delta[2(\nu+1)(R_s/R)^2+1]}{\Gamma(\nu+1)[10-3(\nu+1)\mu\delta(R_s/R)^2]^2}}\right), \tag{44}$$

and for the radius,

$$R^2 = \frac{(\nu+1)g\delta R_s^2}{16(1-\delta)}\left(\sqrt{1+\frac{g\delta}{4(1-\delta)}}-1\right)^{-1}, \tag{45}$$

where

$$\mu = 4\gamma\varepsilon_{\max}^{(-)}/(\nu+1), \quad \delta=\varepsilon/\varepsilon_{\max}^{(-)}, \quad g=4+12\mu+9\mu^2\delta.$$

It is noteworthy that we have a double restriction on ε in this case:

$$\varepsilon_{\min}^{(-)} < \varepsilon < \varepsilon_{\max}^{(-)}, \tag{46}$$

where

$$\varepsilon_{\max}^{(-)} = \frac{5^{5\nu+2}\Gamma(3\nu+1)^2}{4\times 3^{6\nu+4}\Gamma(\nu+1)\Gamma(5\nu+1)}, \tag{47}$$

$$\varepsilon_{\min}^{(-)} = \frac{\varepsilon_{\max}^{(-)}(9\mu^2+28\mu+12)}{6\mu^2} \times \left(\sqrt{1+\frac{16\mu^2(9\mu+5)}{(9\mu^2+28\mu+12)^2}}-1\right). \tag{48}$$

The upper limit is a condition of the existence of a solution, while the lower limit is a condition of stability.

For $\nu=1$ we have $\varepsilon_{\max}^{(-)}=0.099$, so the agreement is relatively good, despite the difference between Eq. (43) and the actual shape of the solution.

To check the analytical predictions we make a numerical calculation. For this purpose we do not take any analytical approximation of f, but use the exact numerical solution $f \equiv U$, $A_0=1$, and $R_0=R_s$ for the calculation of the coefficients J_1–J_6. Assuming that the pulse has no initial chirp $\sigma\sim R' =0$, we can solve Eq. (41) numerically, using the conditions $R'(0)=0$ and $R_0=R_s$. In case of stable solutions the radius must have finite values, oscillating near the minimum of potential (42) (see Fig. 3).

The numerical results give $\varepsilon_{\min}^{(-)}=0.063$ in the case of light "bullets" ($\nu=0$) and $\varepsilon_{\min}^{(-)}=0.085$ in the case of "bright" vortices at $\nu=1$. Let us compare numerical and analytical results. We have $\varepsilon_{\min}^{(-)}/\varepsilon_{\max}^{(-)}|_{\text{num}}=0.46$ and $\varepsilon_{\min}^{(-)}/\varepsilon_{\max}^{(-)}|_{\text{anal}}=0.51$ in the case of $\nu=0$. If $\nu=1$, we obtain $\varepsilon_{\min}^{(-)}/\varepsilon_{\max}^{(-)}|_{\text{num}}=0.62$ and $\varepsilon_{\min}^{(-)}/\varepsilon_{\max}^{(-)}|_{\text{anal}}=0.53$. Thus, the agreement is obvious.

It is seen from Eqs. (13) and (14) that

$$\varepsilon = \frac{8b}{3a}\left(\frac{E_m}{\xi_m}\right)^2 \approx \frac{8b}{3a}\frac{I_e}{I_o},$$

where I_e and I_o are the intensities of the ordinary and extraordinary pulse components, respectively. Starting from (46) we can rewrite the condition of the existence stable two-component "bright" solitons as follows:

$$\frac{3a}{8b}\varepsilon_{\min}^{(-)} < \frac{I_e}{I_o} < \frac{3a}{8b}\varepsilon_{\max}^{(-)}. \tag{49}$$

Substituting here the expressions for a and b, we can rewrite this condition through internal parameters of the medium:

$$\frac{3\chi_{eo}}{4\chi_{ee}}\varepsilon_{\min}^{(-)} < \frac{I_e}{I_o} < \frac{3\chi_{eo}}{4\chi_{ee}}\varepsilon_{\max}^{(-)}. \tag{50}$$

If $\chi_{ee}=0$, we obtain from (50) that the relation I_e/I_o tends to infinity. Therefore, the role of nonlinearity, owned by the USP component, is essential for the stability of these solitons.

V. CONCLUSION

Thus, in present work we predict certain types of three-dimensional two-component optical pulses. The "dark" hole-vortex solitons exist in an area of normal GVD. They have the form of field defects on an infinite background. The quasimonochromatic component has a vortex structure, and the USP component has a "hole" in the center.

In the opposite case of anomalous GVD the existence of fully localized structures is possible. The components may have the form of light "bullets" or "bright" hole-vortex solitons. The last ones also have hole-vortex structure as in the case of normal GVD, but they are localized in all dimensions.

The approximate stability analysis, made with a help of a variational approach, had shown the stability of these structures with respect to self-focusing. The role of quadratic nonlinearity, owned by the USP component, is principal in the arrest of collapse in an area of anomalous GVD. It may be understood as the influence of the extraordinary component on the refraction index of the ordinary component.

However, the question of stability of the structures discussed is still open. This especially relates to the azimuthal instability of vortices with large charges. Full understanding of the formation of these objects and their complete stability analysis can be given in terms of future direct numerical simulations.

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